MATH 2500: LIMIT PROPERTIES

The $\epsilon - \delta$ Definition of Limit:

$\lim_{x\to a} f(x) = L$ means that given any $\epsilon > 0$, there is a $\delta > 0$ so that if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.

Geometrically, we can get $y = f(x)$ 'trapped' between the horizontal lines $y = L - \epsilon$ and $y = L + \epsilon$ by restricting x between the vertical lines $x = a - \delta$ and $x = a + \delta$. The portion of the inequality '$0 < |x-a|$' reminds us **the limit doesn't care about what's actually happening at** $x = a$. Using this definition, we can prove:

Limits are Unique: If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$, then $L_1 = L_2$. Hence we may speak of '*the*' limit of f.

Theorem (Basic Limits):

1. $\lim_{x\to a} b = b$ for any constant, b.
2. $\lim_{x\to a} x = c$

Theorem (Limits Respect Function Arithmetic):[1]

Suppose $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = K$, and let k be any real number. Then:

1. **Scalar Multiple Rule:** $\lim_{x\to a} [kf(x)] = k\left[\lim_{x\to a} f(x)\right] = kL$
2. **Sum / Difference Rule:** $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm K$
3. **Product Rule:** $\lim_{x\to a} [f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right] = LK$
4. **Quotient Rule:** $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{K}$, provided $K \neq 0$.
5. **Power Rule:** $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = L^n$, where n is any natural number ($n = 1, 2, 3, \ldots$)
6. **Function Composition:** If we assume $\lim_{x\to L} g(x) = g(L)$, then $\lim_{x\to a} g(f(x)) = g\left(\lim_{x\to a} f(x)\right) = g(L)$.

Rule of Thumb (Substitution Principle): If $f(a)$ exists, then, *most* of the time, $\lim_{x\to a} f(x) = f(a)$.

Exceptions: Be wary of piecewise-defined functions at 'break' points and and even roots.

Limits of Radicals: If n is odd, the following limit holds for all real numbers a. If n is even, the following holds for all *positive* real numbers a:

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$$

NOTE: For even-indexed radicals, $\lim_{x\to 0} \sqrt[n]{x}$ does not exist since as $x \to 0^-$, $\sqrt[n]{x}$ assumes non-real number values. What happens when $f(c)$ doesn't exist? We can try to algebraically manipulate the function f into an easier function g to deal with, or we can try to bound f between functions with a common limit:

Theorem: If $f(x) = g(x)$ for all x 'near' a, except possibly at $x = c$, and $\lim_{x\to a} g(x) = L$, then $\lim_{x\to a} f(x) = L$.

The Squeeze Theorem: If $g(x) \leq f(x) \leq h(x)$ for all x 'near' a, except possibly at $x = a$, and, additionally: $\lim_{x\to a} g(x) = L$ and $\lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$.

Rule of Thumb (Indeterminate Form): If $\lim_{x\to a} f(x)$ results in the indeterminate form '$\frac{0}{0}$', then there's a *chance* we manipulate the expression $f(x)$ to get the limit to exist or use the Squeeze Theorem.

[1]**NOTE:** All of these properties, including the Squeeze Theorem, hold for one-sided limits and limits at infinity.